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Pricing Insurance Risk: Theory and Practice

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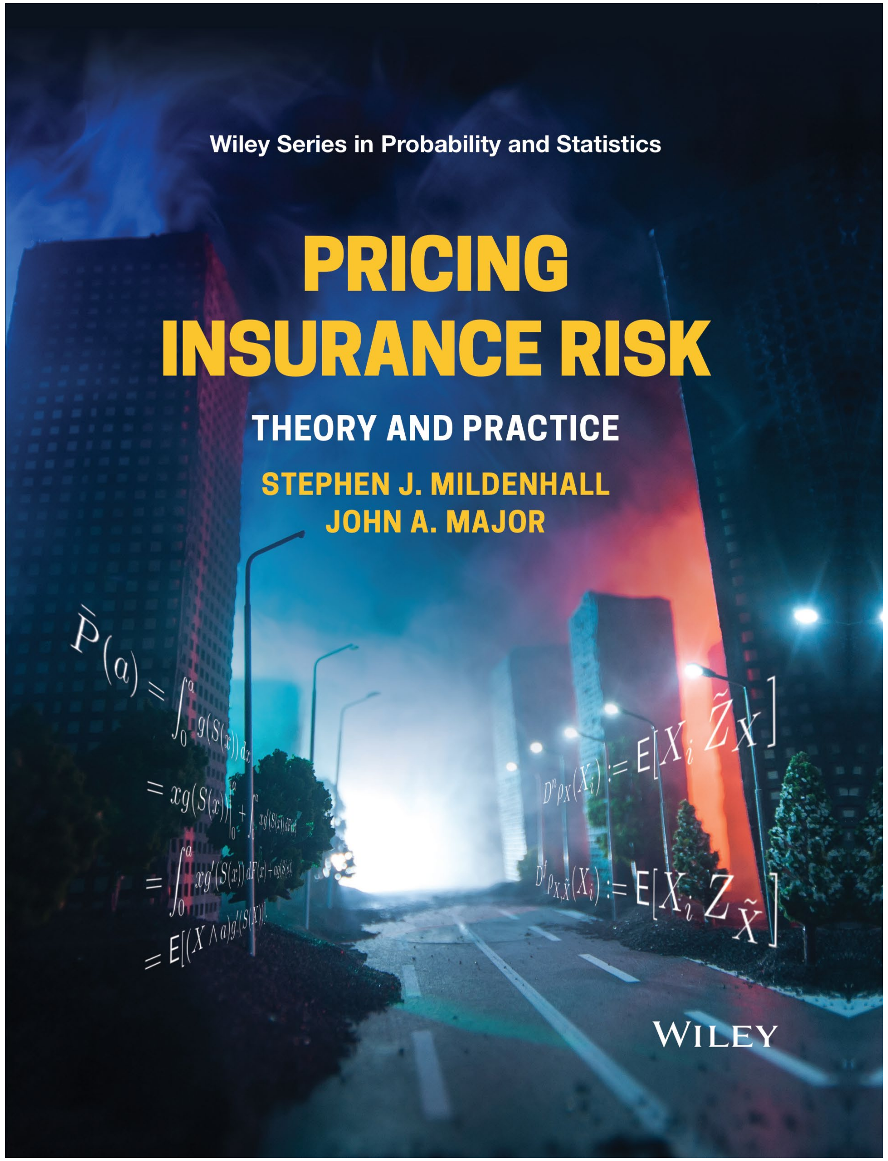
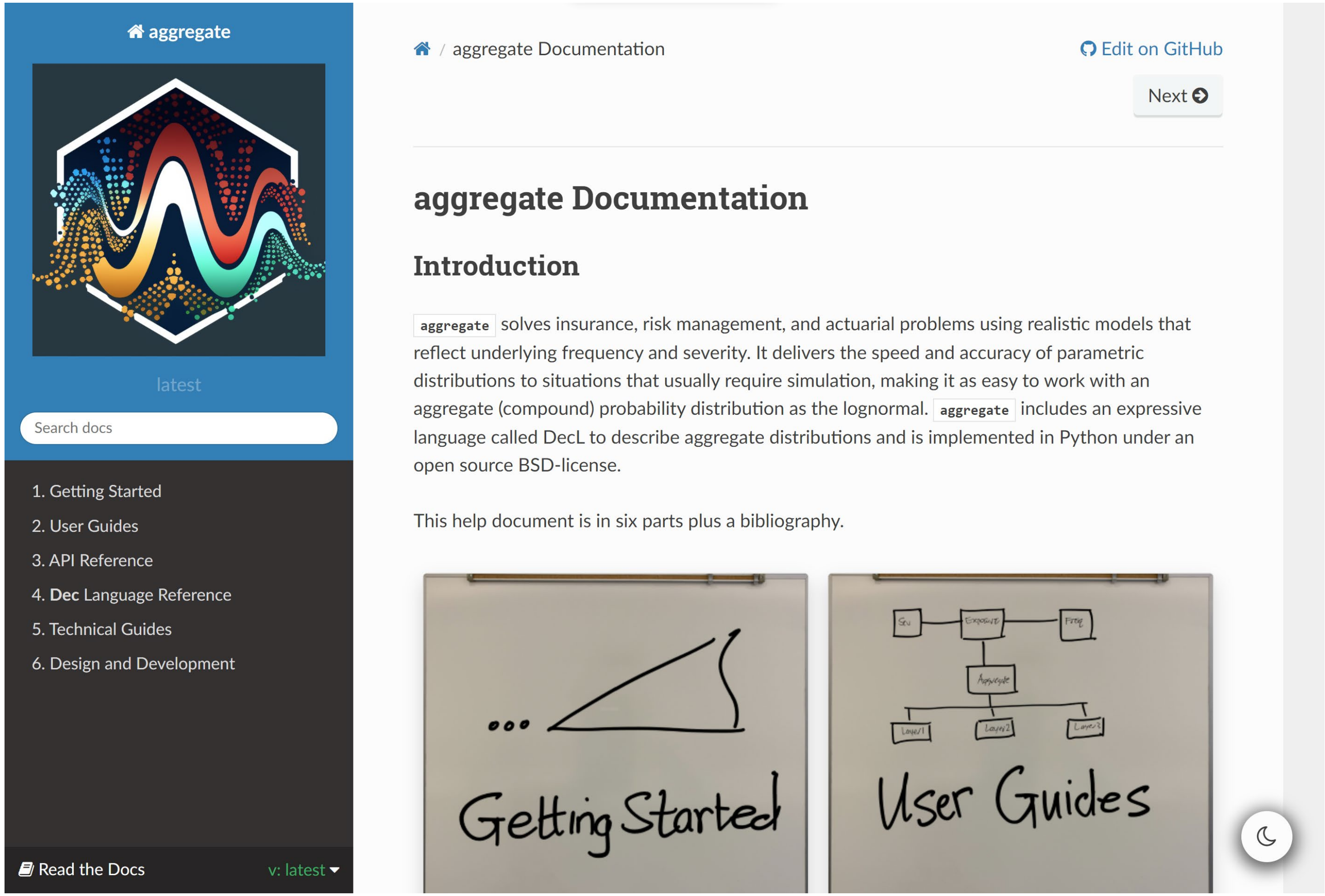


http://www.pricinginsurancerisk.com



https://aggregate.readthedocs.io/en/latest/

Four Themes

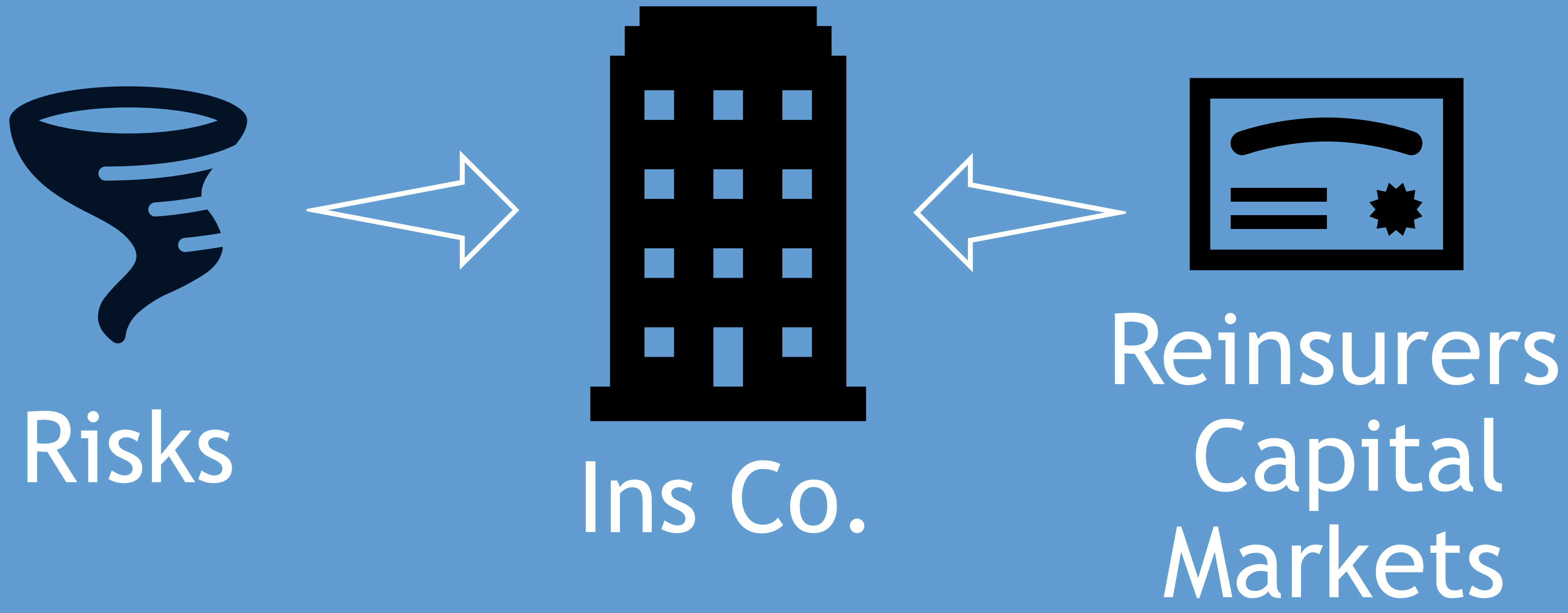
Insurers aggregate contracts specifying contingent cash flows.

Motivation matters: price depends on who initiates the transaction.

Spectral methods reflect motivation and link risk appetite to price.

aggregate Python package provides an open-source implementation.

Ins Co.: One-period insurer, no default*



$t = 0$	Premium →	← Collateral or Capital
$t = 1$	Loss payments ←	→ Residual collateral or assets

* Default is an important but irrelevant complication

Ins Co. $t = 1$ Cash Flows

	X1	X2	X3	X4
0	36	0	29	35
1	40	0	25	35
2	28	0	37	35
3	22	0	43	35
4	33	7	25	35
5	32	8	25	35
6	31	9	25	35
7	45	10	10	35
8	25	40	0	35
9	25	75	0	0

- Cash flows from insurer to each counter-party at $t = 1$
- Ten scenarios, 0-9 (Python…)
- Scenarios equally likely

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
9	25	75	0	0	100

- Payments sum to 100 in every scenario
- No net risk
- Risk-free rate zero
- Starting assets must equal 100
- No net risk margin

- Starting assets funded by amounts paid at t = 0 to purchase each flow

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
9	25	75	0	0	100
EX	31.700	14.900	21.900	31.500	100
CV	0.215	1.545	0.623	0.333	0
Skew	0.456	1.791	-0.369	-2.667	0

- X_1, X_2 appear insurance-like
  - Moderate to high CV
  - Positive skewness
- X_1 non-cat line
  - Attritional payments in all scenarios
  - Moderate CV
- X_2 cat line
  - 40% chance of no payment
  - Extreme CV and skewness

Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total
0	36	0	29	35	100
1	40	0	25	35	100
2	28	0	37	35	100
3	22	0	43	35	100
4	33	7	25	35	100
5	32	8	25	35	100
6	31	9	25	35	100
7	45	10	10	35	100
8	25	40	0	35	100
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EX	31.700	14.900	21.900	31.500	100
CV	0.215	1.545	0.623	0.333	0
Skew	0.456	1.791	-0.369	-2.667	0

- X_3, X_4 capital or reinsurance-like
  - Negative correlation with $X_1 + X_2$
  - Negative skewness
- X_4 return of collateral on a 35 xs 65 aggregate cover
  - Return 35 = no ceded loss
  - Return 0 = limit loss
  - E.g., cat bond
- X_3 residual value
  - Equity

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Ins Co. t = 1 Cash Flows

	X1	X2	X3	X4	total	Gross	Ceded	Net	Financing
0	36	0	29	35	100	36	0	36	64
1	40	0	25	35	100	40	0	40	60
2	28	0	37	35	100	28	0	28	72
3	22	0	43	35	100	22	0	22	78
4	33	7	25	35	100	40	0	40	60
5	32	8	25	35	100	40	0	40	60
6	31	9	25	35	100	40	0	40	60
7	45	10	10	35	100	55	0	55	45
8	25	40	0	35	100	65	0	65	35
9	25	75	0	0	100	100	35	65	0
EX	31.700	14.900	21.900	31.500	100	46.600	3.500	43.100	53.400
CV	0.215	1.545	0.623	0.333	0	0.515			0.322
Skew	0.456	1.791	-0.369	-2.667	0	1.590			-0.788

- X_1 non-cat insurance
- X_2 cat insurance
- X_3 equity residual
- X_4 35 xs 65 reinsurance

- Gross = $X_1 + X_2$
- Ceded = $35 - X_4$
- Net = Gross – Ceded
- Financing = $X_3 + X_4$
- Gross + Financing = 100

Cash Flow Characteristics

Characteristic	Insurance, risk assumption	Financing, risk bearing
t = 0 flow	Fixed inflow	Fixed inflow
t = 1 flow	Contingent outflow	Contingent outflow
Skewness	Positive	Negative
Margin t=0 flow – E[t=1 flow]	Positive to Ins Co.	Negative to Ins Co.
Management	Underwriting / CUO	Reinsurance Finance / CFO
Motivation	**Initiated (bought) by insured**	**Initiated (sold) by insurer**

Motivation is the differentiating characteristic; it is invisible in cash flows

Pricing: $t = 0$ funding

Standard efficient market pricing rule: state-price density

- $P(X) := E[XZ]$
  - X = random variable giving cash flow in each state of the world
  - Z = **state price density**, values 1 in each state, reflects “the market”
    - $Z \geq 0$, $E[Z] = 1$
    - Z aka likelihood ratio, Q-measure, risk-adjusted probabilities (Z x pdf)
  - Work on sample space $\Omega = [0, 1]$ and treat Z as risk-adjusted probabilities
  - E.g., Black-Scholes, CAPM
  - $P(X) = E[X] + \mathrm{cov}(X, Z)$

- Problems with P as a model of insurance pricing
  - Additive: $P(X + Y) = P(X) + P(Y)$, means no diversification benefit
  - Homogeneous: $P(-X) = -P(X)$, rather than positive homogeneous

Standard pricing rules ignore motivation

- Motivation is key: expect distinct prices depending on buyer/seller motivation
  - $A(X)$: **ask** price for X when the buyer (insured) initiates the transaction
  - $B(X)$: **bid** price for X when the seller (insurer) initiates
  - Same X in both cases

- **No arbitrage**: write X at ask and sell $-X$ at bid yields $X - X = 0$, a risk-free portfolio with value zero, hence proceeds $A(X) + B(-X) = 0$. Therefore

  $$A(X) = -B(-X)$$

  $$B(X) = -A(-X)$$

- $-P(-X) = -E[-XZ] = --E[XZ] = E[XZ] = P(X)$: ask price equals bid, no spread

Better alternative: Spectral Pricing Rule (risk measure)

- $\rho(X) := \max \{ E[XZ] \mid Z \geq 0, E[Z] = 1$, and other characteristics, Z in $\mathcal{Z} \}$

- $\rho(X) = E[XZ_X]$ for Z_X in $\mathcal{Z}$, a customized **contact function** state price density
  - Hardy-Littlewood: X and Z_X must be comonotonic (increase together)
  - Z_X measures how much you care about loss of size X

- The set $\mathcal{Z}$ of acceptable Z can be defined from a distortion function g, an increasing, concave function $[0, 1] \to [0, 1]$, by requiring for all $0 \leq s \leq 1$

$$\int_0^s Z(t)dt \leq g(s)$$

E.g., $Z(s) = g'(s)$

Spectral pricing rules have positive bid-ask spreads

- $\rho(X) = \max \{ E[XZ] \mid Z \ge 0, E[Z] = 1$, and other characteristics, Z in $\mathcal{Z} \}$
- If $\mathcal{Z}$ is large enough, then $\rho(X) > E[X]$ because Z_X weights bad (large) outcomes more than small ones; hence, interpret $\rho(X) = A(X)$ as the **ask** price
- $$\begin{aligned} B(X) &= -A(-X) \\ &= -(\max_Z E[-XZ]) \\ &= -(-\min_Z E[XZ]) \\ &= \min_Z E[XZ] \end{aligned}$$
- $B(X) < E[X]$ and spread is positive $A(X) - B(X) = A(X) + A(-X) > A(X - X) = 0$

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Spectral pricing rules have many other nice properties

Pricing rule properties

a) **Monotone**: X ≤ Y implies that ρ(X) ≤ ρ(Y)

b) **Sub-additive**: respects diversification: ρ(X + Y) ≤ ρ(X) + ρ(Y)

c) **Comonotonic additive**: no credit when no diversification. If outcomes X and Y imply same event order, then ρ(X + Y) = ρ(X) + ρ(Y)

d) **Law invariant**: ρ(X) depends only on the distribution of X; no categorical line CoC

A **spectral risk measure** (SRM) ρ(X) is characterized by (a)-(d). They have four representations:

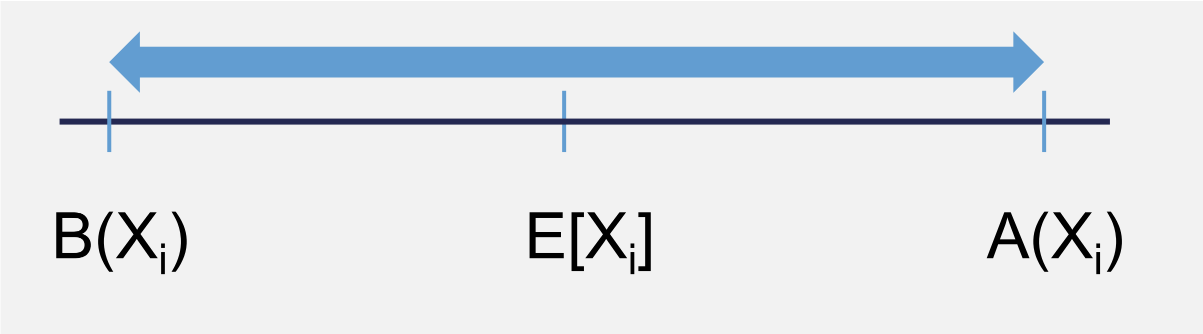
1. Weighted average of VaRs
2. Weighted average of TVaRs
3. Worst over a set of probability scenarios, max { E[XZ] | Z in $\mathcal{Z}_g$}
4. Distorted expected value

$$\rho_g(X) := \int_0^\infty g(S_X(x))\,dx = \mathsf{E}[Xg'(S(X))]$$

See: PIR Theorem 3, p.261

The (Linear) Natural Allocation

- $\rho(X) = E[XZ_X]$
- If $X = X_1 + \ldots + X_n$ it is natural to allocate $E[X_i Z_X]$ to unit i
- Need to be careful Z_X is unique
- In general $E[X_i Z_X] = E[X_i g'(S(X))]$
- Notation: $NA_X(X_i) := E[X_i Z_X]$



- Natural allocation lies between stand-alone bid and ask prices
- X_i comonotonic with X, NA = ask → pure insurance risk
- X_i anti-comon with X, NA = bid → pure financing risk

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Spectral calculations with insurance cash flows

Scenario	X1	X2	X	P	S(X)	g(S)	Q=diff g(S)
3	22	0	22	0.1	0.9	0.974599	0.025401
2	28	0	28	0.1	0.8	0.923257	0.051342
0	36	0	36	0.1	0.7	0.853469	0.069788
1,4,5,6	34	6	40	0.4	0.3	0.433881	0.419588
7	45	10	55	0.1	0.2	0.299491	0.13439
8	25	40	65	0.1	0.1	0.154702	0.144789
9	25	75	100	0.1	0	0	0.154702

EP	31.7	14.9	46.6
EQ	32.31	21.256	53.565
LR	0.9811	0.701	0.87

- EP = loss cost; EQ = technical premium
- Non-cat priced to 98% loss ratio (no expenses)
- Cat priced to 70%
- Overall 87%

- Collapse outcomes by value of X, E[• | X]
- $S(x) = \Pr(X > x)$
- Dual distortion $g(s) = 1 - (1 - s)^{1.59515}$
- Calibrated to 15% return with assets a = 100
- No default
- Z = Q / P
- EP, EQ sum product with P, Q columns

PIR Algos 11.1.1 p.271 and 15.1.1, p.397

Spectral calculations with financing cash flows

Scenario	X3	X4	Financing
3	43	35	78
2	37	35	72
0	29	35	64
1,4,5,6	25	35	60
7	10	35	45
8	0	35	35
9	0	0	0

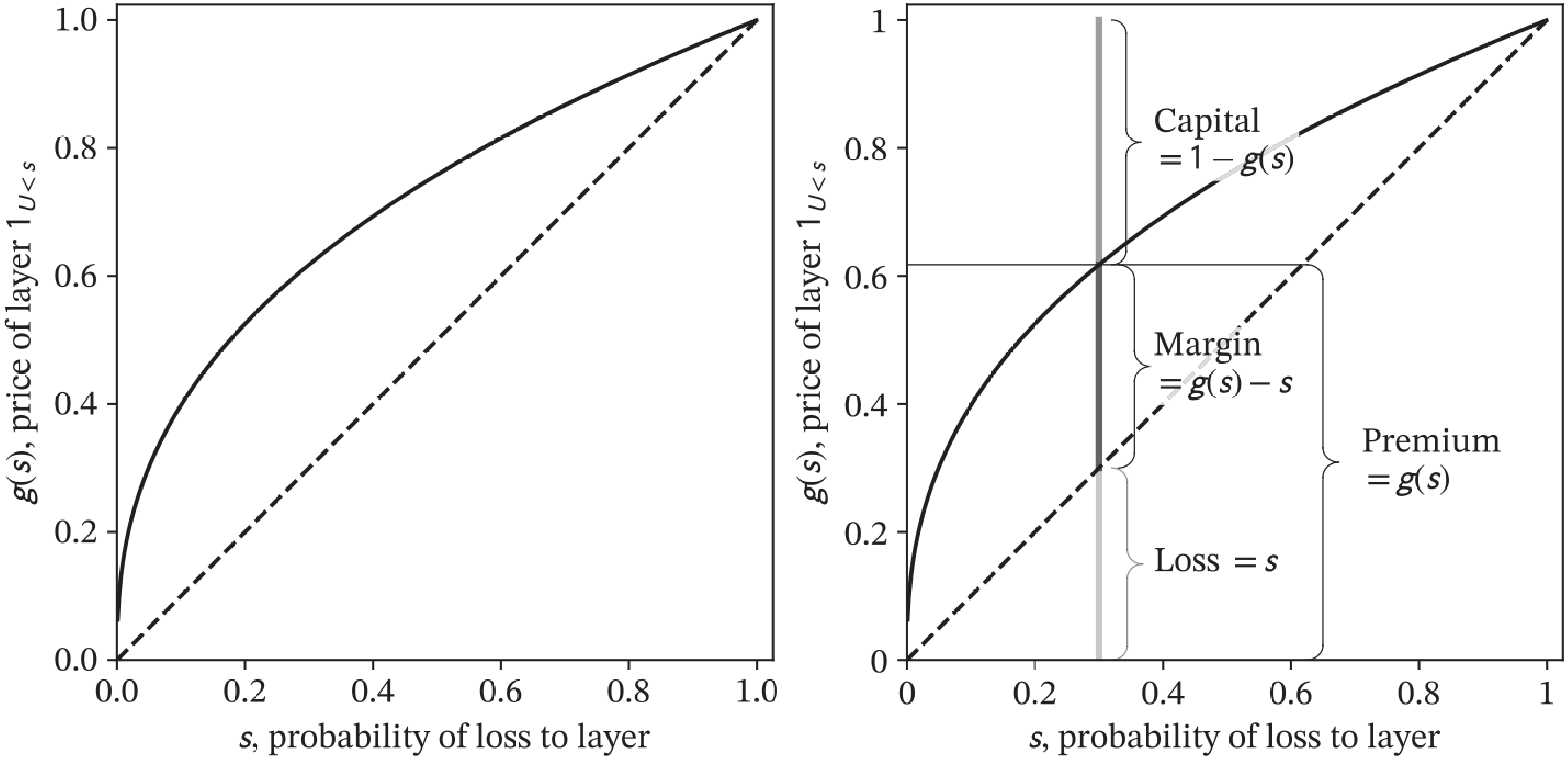
Expected	21.9	31.5	53.4
Price	16.84935	29.58543	46.43478
Return	0.299753	0.064713	0.15

- X_3 equity 30% return
- X_4 agg stop cat bond 6.47% return
- Overall return 15%

- Bid price: sort in descending order
- Expected value of $t = 1$ flow (EP)
- Price is minimum acceptable bid at $t = 0$ for cash flows made at $t = 1$ (EQ)
- Price column also equals $\min_Z E[X_i Z]$
- Return = Expected value / Price – 1
- Achieves 15% overall return
- Implied ceded loss ratio: 64.6%

Financing distinct from asset risk!

Distortion function: g(s) = ask price for Bernoulli 0/1 risk



Graphic: Pricing Insurance Risk, Mildenhall & Major (2022), Wiley

Determining g or $\mathcal{Z}$

Calibrate g to 15% return: five usual suspect distortions

distortion	LR X1	LR X2	LR total
ccoc	102.8%	65.5%	87.0%
ph	101.7%	66.5%	87.0%
wang	100.1%	68.0%	87.0%
dual	98.1%	70.1%	87.0%
tvar	95.7%	72.9%	87.0%

distortion	ROI X3	ROI X4	ROI total
ccoc	15.0%	15.0%	15.0%
ph	21.0%	11.2%	15.0%
wang	25.0%	8.9%	15.0%
dual	30.0%	6.5%	15.0%
tvar	34.9%	4.3%	15.0%

- PIR §11.3 for a description of the constant cost of capital (CCoC), proportional hazard, Wang, dual, and TVaR distortions
- CCoC most sensitive to tail-risk; TVaR most sensitive to body-risk (volatility)
- Sensitives consistent with implied loss ratios (insurance) or returns (financing)

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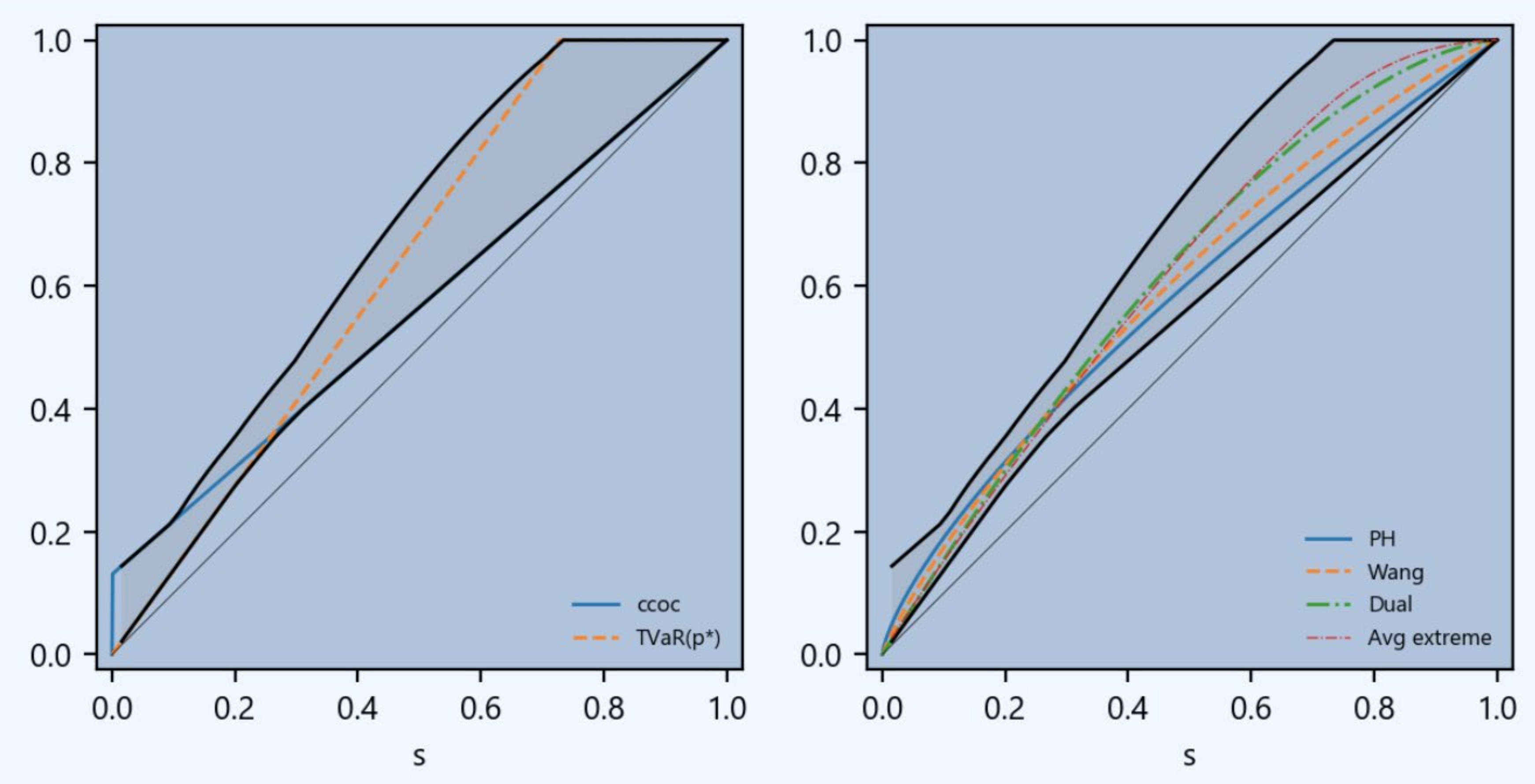
Calibrate g to 15% return: five usual suspect distortions

LR			
unit	X1	X2	total
distortion			
ccoc	102.8%	65.5%	87.0%
ph	101.7%	66.5%	87.0%
wang	100.1%	68.0%	87.0%
dual	98.1%	70.1%	87.0%
tvar	95.7%	72.9%	87.0%

ROI			
unit	X3	X4	total
distortion			
ccoc	15.0%	15.0%	15.0%
ph	21.0%	11.2%	15.0%
wang	25.0%	8.9%	15.0%
dual	30.0%	6.5%	15.0%
tvar	34.9%	4.3%	15.0%

- CCoC: negative margin for X_1, very expensive for cat, X_2
- TVaR: more balanced, positive returns for both lines
- X_4 cat cover price declines with distortion body-centricity
- X_3 equity price increases with distortion body-centricity

Calibrate g to 15% return: five usual suspect distortions



- Shaded area shows all possible distortions
- Left plot: CCoC and TVaR, extreme tail and body sensitivity
- Right plot: PH, Wang, Dual

The Switcheroo

$$X_i \longleftrightarrow E[X_i \mid X]$$

Can exchange X_i and $E[X_i \mid X]$

- $E[X_i \mid X]$ is a random variable: $E[X_i \mid X](\omega) = E[X_i \mid X=X(\omega)]$
- Reduces multi-dimensional problem to one dimension
- $E[X_i Z] = E[E[X_i Z \mid X]] = E[E[X_i \mid X] Z]$
  – Having arranged all Z to be functions of X (**linear** natural allocation)
- Stand-alone price of X_i and $E[X_i \mid X]$ are equal
- Linear natural allocation to X_i and $E[X_i \mid X]$ are equal
- For simulations with distinct X values, $E[X_i \mid X] = X_i$

$E[X_i \mid X]$ and the natural allocation

- Have seen the natural allocation to X_i lies between stand-alone bid and ask prices for X_i, in fact more is true:
  - If $E[X_i \mid X]$ is comonotonic with X, then natural allocation equals $A(E[X_i \mid X])$
    - Pure risk transfer
  - If $E[X_i \mid X]$ is anti-comonotonic with X, then natural allocation equals $B(E[X_i \mid X])$
    - Pure financing

- Easier to meet, test, and see conditions on $E[X_i \mid X]$ than X_i

- If X_i are all thin-tail then $E[X_i \mid X]$ increases with X (Effron's theorem)
  - Ideal insurance situation, most effective diversification

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Decomposing the natural allocation price

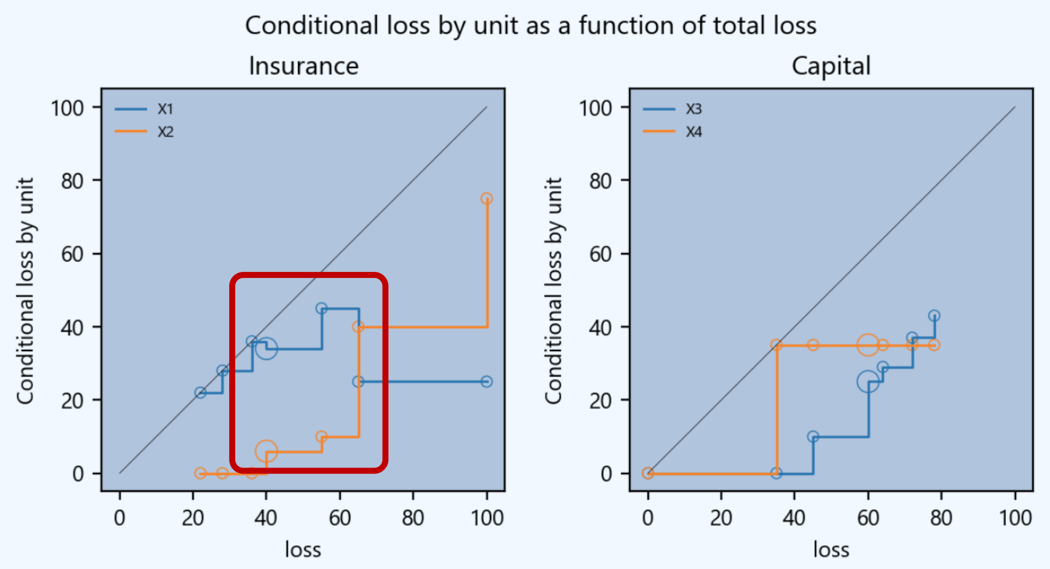
- Can decompose $E[X_i \mid X]$ into $X_i^+ - X_i^-$ where X_i^+, X_i^- are comonotonic with X
- Produces a split $NA(X_i) = E[(X_i^+ - X_i^-)Z] = A(X_i^+) - A(X_i^-)$

  $A(X_i^+)$ = insurance cost with a positive margin

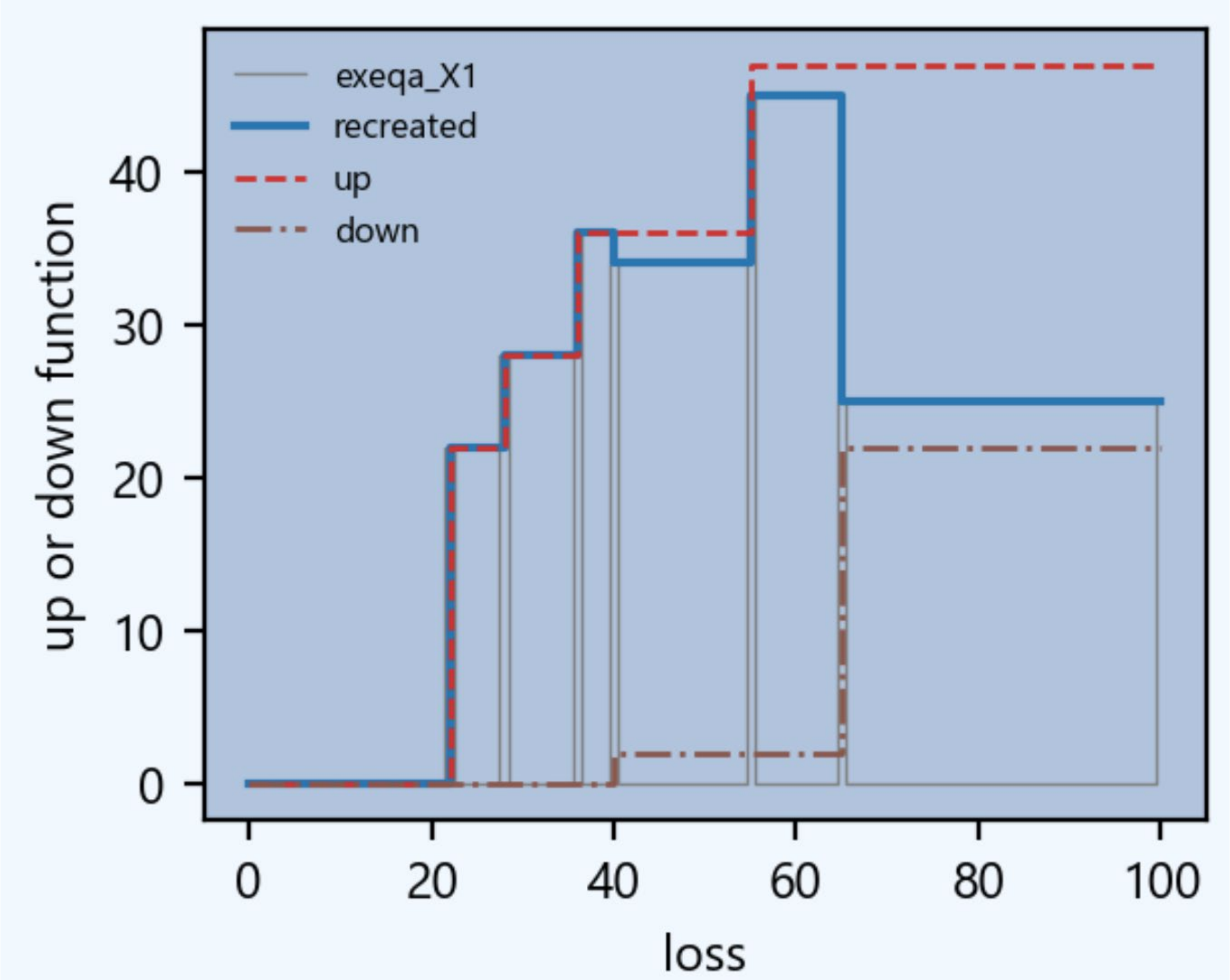
  $-A(X_i^-)$ = financing benefit from selling the capital benefit of X_i, negative margin
- Applies to X_1 but not X_2 which is comonotonic with X

unit		lna	sa	proj_sa	up	down	umd
X1	el	31.700	31.700	31.700	37.100	5.400	31.700
	bid	31.621	28.999	29.273	34.288	2.667	31.621
	ask	32.310	34.288	34.084	40.006	7.697	32.310

Key lna: linear natural allocation; sa = stand-alone, proj_sa = stand-alone $E[X_i \mid X]$; up=X^+, down=X^-, umd = up price minus down; Insurance (up) margin = 40.0 – 37.1 = 2.9; financing (down) offset = 7.7 – 5.4 = 2.3, net 2.9 – 2.3 = 0.6; net lna margin = 32.3 – 31.7 = 0.6.



Decomposing the natural allocation price (details)



- Decomposition is not always possible in theory, but it is in practice.
- exeqa_X1 = E[X1 | X]

Applications

- Diversifying cat: Chile quake, Japan, Australia
- Diversifying cat is not comonotonic with total losses
  → has financing offset which rationalizes a lower margin
- Management: underwriting departments selling capital is fraught

Practical demonstration

Conclusions

1. SRMs are a practical but under-specified pricing tool.

2. Parameterize to premium and financing data, reveals risk appetites.

3. Decompose premiums into insurance and financing parts.

4. Implement calculations using aggregate Python library.